

Wiener Filtering of the *COBE* DMR Data.

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Abstract.

We derive an optimal linear filter to suppress the noise from the *COBE* DMR sky maps for a given power spectrum. We then apply the filter to the first-year DMR data, after removing pixels within 20° of the Galactic plane from the data. The filtered data have uncertainties 12 times smaller than the noise level of the raw data. We use the formalism of constrained realizations of Gaussian random fields to assess the uncertainty in the filtered sky maps. In addition to improving the signal-to-noise ratio of the map as a whole, these techniques allow us to recover some information about the CMB anisotropy in the missing Galactic plane region. From these maps we are able to determine which hot and cold spots in the data are statistically significant, and which may have been produced by noise. In addition, the filtered maps can be used for comparison with other experiments on similar angular scales.

Introduction.

The discovery by the *COBE* DMR of large-scale anisotropy in the cosmic microwave background (CMB) has ushered in a new, more quantitative era in the study of microwave background anisotropy (Smoot *et al.* 1992). The statistical properties of the fluctuations in the DMR sky maps provide an important estimate of the amplitude of cosmological fluctuations on very large scales, which are impossible to observe in any other way.

Most analyses of the DMR sky maps have focused on the problem of estimating statistical properties of the anisotropy such as the power spectrum (*e.g.* Wright *et al.* 1992, Seljak & Bertschinger 1993, Wright *et al.* 1994), or on testing the hypothesis that the anisotropy obeys Gaussian statistics (*e.g.* Hinshaw *et al.* 1994, Smoot *et al.* 1994). In addition, statistical comparisons have been made between the DMR sky maps and data from other experiments (Watson *et al.* 1993, Ganga *et al.* 1993). Less attention has been paid to studying the spatial properties of the sky maps themselves to determine, for example, which hot and cold spots on the maps are likely to be real, and which are dominated by noise. Such an analysis has the potential to be quite useful in comparing the DMR results to other experiments that probe comparable angular scales, such as the FIRS and Tenerife experiments (Meyer *et al.* 1991, Hancock *et al.* 1994).

There are two major difficulties in studying the DMR maps. The signal-to-noise ratio in each pixel is quite low, and pixels near the Galactic plane are contaminated by Galactic emission. These problems place strict limits on the accuracy with which the true structure can be recovered (Bunn, Hoffman, & Silk 1993). In this *Letter* we apply the techniques of Wiener filtering and constrained realizations in an attempt to mitigate these problems. We assume the correctness of the canonical theory of large-scale CMB anisotropy: that the anisotropy forms a Gaussian random field with a power spectrum that is a power law in spatial wavenumber k . We then remove the pixels that are presumed to be contaminated, and apply an optimal linear filter to the data in an attempt to clean up the noise and see the underlying structure. The technique of constrained realizations helps quantify the uncertainties associated with this method.

The Wiener Filter and Constrained Realizations.

The Wiener filtering described in this work is similar to a technique applied recently to galaxy catalogues (Lahav *et al.* 1994). The formalism for making constrained realizations of Gaussian random fields has also been discussed elsewhere (Hoffman & Ribak 1991,

Gannon & Hoffman 1993), as has the connection between this formalism and the Wiener filter (Zaroubi *et al.* 1994). In this section we simply present these techniques in a language appropriate for analyzing the DMR data.

Let us begin by writing the true anisotropy ΔT as an expansion in spherical harmonics*:

$$\Delta T(\hat{\mathbf{r}}) = \sum_{l=2}^{\infty} \sum_{m=-l}^l a_{lm} Y_{lm}(\hat{\mathbf{r}}). \quad (1)$$

We assume that the anisotropy forms a Gaussian random field: each coefficient a_{lm} is an independent Gaussian random variable of zero mean (*e.g.* Bond & Efstathiou 1987). The angular power spectrum of this Gaussian random field is given by

$$C_l = \langle a_{lm}^2 \rangle, \quad (2)$$

where the angle brackets denote an ensemble average.

Each DMR pixel contains a measurement of the beam-smoothed temperature at a particular point: if we denote the i th data point by D_i , then

$$D_i = (\Delta T \star B)(\hat{\mathbf{r}}_i) + n_i. \quad (3)$$

Here B denotes the DMR beam pattern, the star represents a convolution, $\hat{\mathbf{r}}_i$ is a unit vector in the direction of the i th pixel, and n_i is the noise in the pixel. The convolution is performed quite simply in spherical harmonic space: the coefficients of the spherical harmonic expansion of $(\Delta T \star B)$ are given by $a'_{lm} = B_l a_{lm}$, where

$$B_l = 2\pi \int_0^\pi B(\theta) P_l(\cos \theta) \sin \theta d\theta, \quad (4)$$

* Throughout this *Letter*, the symbol Y_{lm} will denote a real-valued spherical harmonic. The real-valued spherical harmonics are simply the conventional spherical harmonics with complex exponentials replaced by ordinary trigonometric functions:

$$Y_{lm}(\theta, \phi) = \begin{cases} Y_{lm}^{(conv)}(\theta, \phi) & \text{for } m = 0 \\ Y_{lm}^{(conv)}(\theta, 0) \sqrt{2} \cos m\phi & \text{for } 1 \leq m \leq l \\ Y_{lm}^{(conv)}(\theta, 0) \sqrt{2} \sin m\phi & \text{for } -l \leq m \leq -1 \end{cases}.$$

These functions satisfy the usual orthonormality condition $\int Y_{lm}(\theta, \phi) Y_{l'm'}(\theta, \phi) d\Omega = \delta_{ll'} \delta_{mm'}$.

and P_l is a Legendre polynomial. The beam pattern is often approximated by a Gaussian, but we use the actual DMR beam pattern given in Wright *et al.* (1994).

The noise is presumed to consist of independent Gaussian random numbers with known variances σ_i^2 . The assumption that the noise in different pixels is uncorrelated has been recently confirmed by Lineweaver *et al.* (1994). This represents one significant difference between the present work and the analysis of galaxy catalogues in Lahav *et al.* (1994). The shot noise in the galaxy distribution is not independent from pixel to pixel, because it has been smoothed along with the signal. The DMR instrumental noise enters the data after the smoothing by the beam has already taken place.

Pixels near the Galactic plane are presumed to be contaminated. Even though we work with the “Reduced Galaxy” linear combination of sky maps (Smoot *et al.* 1992), the pixels within 20° of the Galactic plane are excised from the data set. The number of pixels left after this operation is $N_{pix} = 4038$.

If we assume that we know the power spectrum C_l , we can apply a Wiener filter to reconstruct ΔT from the data. We begin by generating initial estimates b_{lm} of the spherical harmonic coefficients:

$$b_\mu = \sum_{i=1}^{N_{pix}} w_i Y_\mu(\hat{\mathbf{r}}_i) D_i, \quad (5)$$

where the weights are given by $w_i = 1/\sigma_i^2$. Here and hereafter, Greek indices denote pairs (l, m) . (The estimated coefficients b_μ are not normalized properly, but the overall normalization is irrelevant for our purposes.) The monopole and dipole terms are removed from the DMR sky maps before these recovered coefficients are computed.

The estimated spherical harmonic coefficients b_μ are related to the true coefficients a_μ as follows:

$$b_\mu = \sum_{\nu} W_{\mu\nu} B_\nu a_\nu + \sum_{i=1}^{N_{pix}} w_i Y_\mu(\hat{\mathbf{r}}_i) n_i, \quad (6)$$

where

$$W_{\mu\nu} = \sum_{i=1}^{N_{pix}} w_i Y_\mu(\hat{\mathbf{r}}_i) Y_\nu(\hat{\mathbf{r}}_i). \quad (7)$$

The matrix W describes the coupling that is introduced by incomplete sky coverage between the different spherical harmonics (Peebles 1980, Scharf *et al.* 1992, Lahav *et al.* 1994).

We want to apply a linear filter to these coefficients to get a vector of new coefficients \vec{c} that is as close as possible to the original coefficients \vec{a} , in the sense of least squares.

That is to say, we want to define a set of coefficients

$$\vec{c} = F\vec{b}, \quad (8)$$

where the matrix F has been chosen to minimize

$$\Delta = \langle |\vec{c} - \vec{a}|^2 \rangle = \left\langle \sum_{\mu} (c_{\mu} - a_{\mu})^2 \right\rangle \quad (9)$$

By substituting equations (5), (6) and (8) into this expression, and setting $\partial\Delta/\partial F_{\mu\nu} = 0$, we find the appropriate filter F :

$$F = CB(WBCB + 1)^{-1}. \quad (10)$$

In the above equation W is the matrix with elements $W_{\mu\nu}$, and B and C are diagonal matrices representing the power spectrum and the beam pattern: $C_{\mu\nu} = C_l\delta_{\mu\nu}$ and $B_{\mu\nu} = B_l\delta_{\mu\nu}$, where l is the multipole number corresponding to μ .

Once the Wiener-filtered coefficients \vec{c} have been determined, the reconstructed temperature anisotropy at any point on the sky is simply

$$\Delta T_W(\hat{\mathbf{r}}) = \sum_{\mu} c_{\mu} Y_{\mu}(\hat{\mathbf{r}}). \quad (11)$$

Of course, we do not expect the temperature at any particular point on the sky to match the Wiener reconstructed value exactly. We can assess the expected fluctuations about the reconstructed map with the formalism of constrained realizations. In deriving the Wiener filter, we assumed that we knew only the power spectrum C_l (*i.e.*, the variances of the a_{lm} 's). We have not yet used our additional assumption that the probability distribution function for the a_{lm} 's is Gaussian. With this assumption, we can use the formalism of conditional probability to compute the probability of any particular realization $\Delta T(\hat{\mathbf{r}})$ being the true anisotropy: if the coefficients of the spherical harmonic expansion of ΔT are a_{lm} , then the probability of ΔT being the true anisotropy, given the data D , is

$$\begin{aligned} p(\Delta T|D) &\propto p(\Delta T) p(D|\Delta T) \\ &\propto \exp(-\frac{1}{2}\vec{a}^T C^{-1}\vec{a}) \exp\left(-\frac{1}{2} \sum_{i=1}^{N_{pix}} \frac{(D_i - (\Delta T \star B)(\hat{\mathbf{r}}_i))^2}{\sigma_i^2}\right). \end{aligned} \quad (12)$$

The first term is simply the probability of getting the realization ΔT given the power spectrum, and the second term is the probability of getting the observed data given the

underlying field ΔT . If we expand ΔT in terms of the a_{lm} 's, and complete the square in the exponential, this expression becomes

$$p(\Delta T|D) \propto \exp\left(-\frac{1}{2}(\vec{a} - \vec{c})^T F^{-1}(\vec{a} - \vec{c})\right). \quad (13)$$

This is a Gaussian probability distribution with mean \vec{c} and covariance matrix F (Rybicki & Press 1992). This probability distribution takes its maximum value when $\Delta T = \Delta T_W$.

From this information we can easily compute the variance of $\Delta T(\hat{\mathbf{r}})$ about the mean value $\Delta T_W(\hat{\mathbf{r}})$, which we can interpret as the uncertainty in the Wiener reconstructed anisotropy at the point $\hat{\mathbf{r}}$. In addition, we can make constrained realizations of the anisotropy ΔT with the probability distribution (13).

Results.

We applied the Wiener filter to the DMR first-year ‘‘Reduced Galaxy’’ sky map, shown in Figure 1. We made three different choices for the power spectrum, corresponding to power-law indices $n = 0.5, 1, 1.5$ for the primordial matter power spectrum. The normalization in each case was given by the correlation function analysis of Seljak & Bertschinger (1993): The quadrupole amplitude $Q_{rms-PS} = 15.7 \exp(0.46(1-n)) \mu\text{K}$. The second-year DMR data are consistent with this amplitude, although the data prefer a slightly lower normalization (Bennett *et al.* 1994)

Given n and Q_{rms-PS} , the power spectrum C_l is

$$C_l = \left(\frac{4\pi Q_{rms-PS}^2}{5}\right) \frac{\Gamma\left(\frac{2l+n-1}{2}\right) \Gamma\left(\frac{9-n}{2}\right)}{\Gamma\left(\frac{2l+5-n}{2}\right) \Gamma\left(\frac{3+n}{2}\right)}. \quad (14)$$

(Bond & Efstathiou 1987).

The filtered sky maps for these three power spectra are shown in Figure 2. Multipoles with $l \leq 30$ were included in the filtering, and the final results were quite insensitive to variations in this cutoff. The insensitivity to the cutoff results from the high noise level in the data: modes with large l are dominated by noise, and so are highly suppressed by the Wiener filter. This is in contrast with the results of Lahav *et al.* (1994) for the galaxy distribution, which have a higher signal-to-noise ratio, and are therefore more sensitive to the cutoff.

The three maps show similar qualitative features, but the maps with larger values of the spectral index n have more small-scale structure. This is not surprising: it simply

reflects the fact that the assumed power spectrum C_l has more small-scale power for large n .

We know that the difference between the actual anisotropy and the Wiener-filtered map is a Gaussian random field with covariance matrix F . The uncertainty $U(\hat{\mathbf{r}})$ in the reconstructed map at any particular point $\hat{\mathbf{r}}$ on the sky is simply the square root of the variance of this field at $\hat{\mathbf{r}}$. The r.m.s value of $U(\hat{\mathbf{r}})$ over the region outside of the galactic cut is $20\,\mu\text{K}$, which is lower than the $240\,\mu\text{K}$ r.m.s. pixel noise in the raw data. (This is what we mean when we say that the Wiener filter “suppresses noise.”) However, the filtering process also suppresses some of the signal. In order to compare the filtered maps to the raw data, we define a signal-to-noise ratio for the filtered maps as $\Delta T_W(\hat{\mathbf{r}})/U(\hat{\mathbf{r}})$.

Figure 3 shows contour plots of this signal-to-noise ratio for the three filtered maps. For the Harrison-Zel’dovich map, 308 of the 4038 pixels that lie outside of the 20° Galactic cut have signal-to-noise ratios greater than 2. This is a considerable improvement over the 211 pixels in the raw data that have signal-to-noise ratios greater than 2. Since no data were used from within the Galactic cut region, the area near the Galactic equator is reconstructed with somewhat lower significance.

The signal-to-noise map in Figure 3 contains no information about correlations between the fluctuations at different points in the sky: it was made using only the diagonal part of the covariance matrix. We can get a more complete picture of the expected fluctuations about the Wiener-filtered map by making constrained realizations. Figure 4 shows three constrained realizations of the CMB anisotropy. These maps were made with the assumption of a Harrison-Zel’dovich power spectrum, and have been smoothed with the *COBE* beam pattern. The constrained realizations show a number of features that persist from map to map. In particular, the hot spots near the Galactic coordinates $(l, b) = (275^\circ, -36^\circ)$ and $(55^\circ, 65^\circ)$ appear highly significant, as does the large cold region around $(260^\circ, 50^\circ)$.

Conclusions.

Wiener filtering is a promising tool for the analysis of CMB sky maps. Filtering provides a significant improvement in the signal-to-noise ratio in the regions covered by the raw data, and allows some information to be reconstructed about the anisotropy within the Galactic cut region. It is possible to identify several hot and cold spots in the map which carry high statistical significance.

This reduction in noise is not without a price. In order to apply the Wiener filter, one needs to assume a power spectrum. The cleaned data therefore depend on more assumptions than do the raw data. However, the maps do not undergo great qualitative changes as one varies the slope of the power spectrum over a wide range of reasonable values.

The expected fluctuations about the Wiener filtered maps are assessed with constrained realizations. This technique can be used to perform Monte-Carlo simulations of the CMB for comparison with experiments on similar angular scales such as the Tenerife (Hancock *et al.* 1994) and FIRS (Ganga *et al.* 1993) experiments. In particular, Hancock *et al.* (1994) have reported a significant fluctuation at a particular location on the sky. Due to the differencing technique of the Tenerife experiment and the difference in angular scales, it is fruitless to try to identify this feature by eye on the filtered DMR maps; however, a quantitative comparison of the Tenerife and FIRS data sets with the filtered DMR maps may prove revealing. This approach makes use of both amplitude and phase information in the two data sets, and is therefore potentially more powerful than power-spectrum estimation techniques, which in general throw away the phase information.

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Figure Captions

Figure 1. The DMR Reduced Galaxy sky map is shown. The upper panel shows the raw data. The lower panel is the data after smoothing with a 7° FWHM Gaussian. Both maps are Aitoff projection, with the North Galactic pole $b = 90^\circ$ at the top and the Galactic center $(l, b) = (0^\circ, 0^\circ)$ at the center of the plot. Longitude increases from right to left. The range of temperatures covered in the map is $(-1000 \mu\text{K}, 1100 \mu\text{K})$.

Figure 2. The Wiener-filtered DMR Reduced Galaxy maps are shown for three different choices of initial power spectrum. The spectral index n and quadrupole normalization Q_{rms-PS} take values $(n, Q_{rms-PS}) = (1.5, 12.5 \mu\text{K}), (1., 15.7 \mu\text{K}), (0.5, 19.8 \mu\text{K})$ in the three panels. The range of temperatures covered in the map is $(-60 \mu\text{K}, 65 \mu\text{K})$. The maps are in Galactic coordinates as in Figure 1.

Figure 3. Contour plots are shown of the signal-to-noise ratio in the Wiener-filtered maps. The contours show regions in which the reconstructed temperature ΔT_W is 1, 2, and 3 times the noise level. Dotted contours denote regions where $\Delta T_W < 0$. The maps are in Galactic coordinates as in Figure 1.

Figure 4. Three constrained realizations of the DMR sky map are shown. The maps were made with the assumption of a Harrison-Zel'dovich ($n = 1$) power spectrum, including multipoles up to $l_{max} = 30$. The maps shown were smoothed with the DMR beam pattern. The range of temperatures covered in the maps is $(-122 \mu\text{K}, 106 \mu\text{K})$. The maps are in Galactic coordinates as in Figure 1.

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